

ON COMPOSITION OF BAIRE FUNCTIONS

OLENA KARLOVA AND VOLODYMYR MYKHAYLYUK

ABSTRACT. We study the maps between topological spaces whose composition with Baire class α maps also belongs to the α 'th Baire class and give characterizations of such maps.

1. INTRODUCTION

In this paper we study right and left Baire compositors and show that under some restrictions on the domain and the range the property of being the right Baire-one compositor is equivalent to many other function properties such as piecewise continuity, G_δ -measurability, B_1 -stability, while left compositors are exactly continuous maps.

By definition, for an ordinal $\alpha \in [0, \omega_1)$ a map $f : X \rightarrow Y$ between topological spaces is the *right (left) B_α -compositor for a class \mathcal{C} of topological spaces* if for any topological space $Z \in \mathcal{C}$ and a map $g : Y \rightarrow Z$ (respectively, $g : Z \rightarrow X$) of the α 'th Baire class the composition $g \circ f : X \rightarrow Z$ (respectively, $f \circ g : Z \rightarrow Y$) also belongs to the α 'th Baire class. Such maps for $X = Y = \mathbb{R}$, $\mathcal{C} = \{\mathbb{R}\}$ and $\alpha = 1$ were introduced and studied by Dongsheng Zhao in [22], where he proved the following result (here and throughout the paper $\mathbb{R}^+ = (0, +\infty)$).

Theorem A. *For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the following conditions are equivalent:*

- (1) *f is G_δ -measurable (i.e., the set $f^{-1}(V)$ is G_δ for any open set $V \subseteq \mathbb{R}$);*
- (2) *f is the right B_1 -compositor;*
- (3) *for any Baire-one function $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^+$ there exists a function $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ such that*

$$(1.1) \quad |x - y| < \min\{\delta(x), \delta(y)\} \implies |f(x) - f(y)| < \min\{\varepsilon(f(x)), \varepsilon(f(y))\}.$$

Moreover, it was introduced the notion of a k -continuous function in [22]. Namely, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *k -continuous* if for any function $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^+$ there exists a function $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ which satisfies (1.1) for all $x, y \in \mathbb{R}$. Obviously, Theorem A implies that every k -continuous function is the right B_1 -compositor. In this connection D. Zhao posed the following question [22, p. 548].

Question 1. *Is every right B_1 -compositor $f : \mathbb{R} \rightarrow \mathbb{R}$ a k -continuous function?*

The positive answer to this question was given independently in [2] and [14]. Observe that crucial auxiliary results were a Jayne–Rogers theorem [8, Theorem 1] in [2] and a Banach–Bokalo theorem [1, Theorem 8.1] in [14] on the equivalence of the G_δ -measurability of a function to the piecewise continuity.

Our goal is to obtain a characterization of the right Baire compositors between arbitrary metrizable (non-separable) spaces and the following theorem (see Theorem 16) is our first main result.

Theorem B. *Let (X, d_X) be a metric space, (Y, d_Y) be a metric space and $f : X \rightarrow Y$ be a map. Consider the following conditions:*

- (1) *f is of the first stable Baire class (i.e., there exists a sequence of continuous maps $f_n : X \rightarrow Y$ such that for every $x \in X$ there is $k \in \mathbb{N}$ with $f_n(x) = f(x)$ for all $n \geq k$);*
- (2) *f is piecewise continuous (i.e., there exists a countable closed cover \mathcal{F} of X such that $f|_F$ is continuous for every $F \in \mathcal{F}$);*
- (3) *for any function $\varepsilon : Y \rightarrow \mathbb{R}^+$ there exists a function $\delta : X \rightarrow \mathbb{R}^+$ such that for all $x, y \in X$*

$$(1.2) \quad d_X(x, y) < \min\{\delta(x), \delta(y)\} \implies d_Y(f(x), f(y)) < \min\{\varepsilon(f(x)), \varepsilon(f(y))\};$$

- (4) *for any function $\varepsilon : Y \rightarrow \mathbb{R}^+$ of the first Baire class there exists a function $\delta : X \rightarrow \mathbb{R}^+$ such that (1.2) holds for all $x, y \in X$;*
- (5) *f is the right B_1 -compositor for the class of all metrizable connected and locally path-connected spaces;*
- (6) *f is G_δ -measurable and σ -discrete (which means that there exists a family $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ of subsets of X , which is called a base for f , such that for any open set $V \subseteq Y$ there is a subfamily $\mathcal{B}_V \subseteq \mathcal{B}$ with $f^{-1}(V) = \bigcup \mathcal{B}_V$ and each family \mathcal{B}_n is discrete in X).*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Leftrightarrow (6)$. If X is a hereditarily Baire space, then $(6) \Rightarrow (2)$. If, moreover, Y is a path-connected space and $Y \in \sigma\text{AE}(X)$, then all the conditions (1)–(6) are equivalent.

We also prove the following result (see Theorem 12) which serves as an important tool in the proof of Theorem B.

Theorem C. Let (X, d_X) be a hereditarily Baire metric space and (Y, d_Y) be a metric space. For a map $f : X \rightarrow Y$ the following conditions are equivalent:

- (1) f is a σ -discrete map with a base which consists of F_σ -subsets of X ;
- (2) f belongs to the first Lebesgue class (i.e., the set $f^{-1}(V)$ is F_σ in X for any open set $V \subseteq Y$);
- (3) f is barely continuous (i.e., for each non-empty closed subset $F \subseteq X$ the restriction $f|_F$ has a continuity point);
- (4) for any $\varepsilon > 0$ there exists a Baire-one function $\delta : X \rightarrow \mathbb{R}^+$ such that for all $x, y \in X$

$$(1.3) \quad d_X(x, y) < \min\{\delta(x), \delta(y)\} \implies d_Y(f(x), f(y)) < \varepsilon.$$
- (5) for any $\varepsilon > 0$ there exists a function $\delta : X \rightarrow \mathbb{R}^+$ such that (1.3) holds for all $x, y \in X$.

Notice that the equivalence of conditions (1) and (2) in Theorem C was established in [6] in the case of absolutely analytic (in particular, complete) metric space X . The equivalence of conditions (2) and (5) was firstly obtained in [17] for Polish spaces X and Y (see also [4] for an alternative proof). Condition (5) for functions $f : \mathbb{R} \rightarrow \mathbb{R}$ was also studied in [18, 19, 20].

Finally, the next our theorem (see Theorem 20) gives a characterization of left Baire compositors.

Theorem D. Let X be a T_1 -space, Y be a perfectly normal space, $f : X \rightarrow Y$ be a map and $\alpha \in [1, \omega_1)$. If one of the following conditions holds:

- (i) $\alpha = 1$ and X is a connected and locally path-connected metrizable space, or
- (ii) $\alpha > 1$ and X is a first countable space such that for any finite sequence U_1, \dots, U_n of open subsets of X there exists a continuous map $\varphi : [1, n] \rightarrow X$ with $\varphi(i) \in U_i$ for every $i \in \{1, n\}$,

then the following conditions are equivalent:

- (1) f is continuous;
- (2) f is the left B_α -compositor.

Observe that similar characterization was proved in [3] for $X = Y = \mathbb{R}$ and $\alpha = 1$.

2. TERMINOLOGY AND NOTATIONS

Let X, Y be topological spaces. By $C(X, Y)$ we denote the collection of all continuous maps between X and Y . A sequence $(f_n)_{n=1}^\infty$ of maps $f_n : X \rightarrow Y$ is *stably convergent* to a map $f : X \rightarrow Y$ on X (we denote this fact by $f_n \xrightarrow{\text{st}} f$) if for every $x \in X$ there exists $k \in \mathbb{N}$ such that $f_n(x) = f(x)$ for all $n \geq k$. If $A \subseteq Y^X$, then the symbols \overline{A}^{st} and \overline{A}^{p} stands for the collection of all stable and pointwise limits of sequences of maps from A , respectively.

We inductively define Baire classes and stable Baire classes as follows: let

$$B_0(X, Y) = B_0^{\text{st}}(X, Y) = C(X, Y)$$

and for each ordinal $\alpha \in (0, \omega_1)$ let $B_\alpha(X, Y)$ be the family of all maps of the α 'th Baire class and $B_\alpha^{\text{st}}(X, Y)$ be the family of all maps of the α 'th stable Baire class which defined by the rules

$$B_\alpha(X, Y) = \overline{\bigcup_{\beta < \alpha} B_\beta(X, Y)}^{\text{p}} \quad \text{and} \quad B_\alpha^{\text{st}}(X, Y) = \overline{\bigcup_{\beta < \alpha} B_\beta^{\text{st}}(X, Y)}^{\text{st}}.$$

The *right B_α -compositor* (for a class \mathcal{C} of topological spaces) is a map $f : X \rightarrow Y$ such that for any space Z (for any $Z \in \mathcal{C}$) and a map $g \in B_\alpha(Y, Z)$ the composition $g \circ f : X \rightarrow Z$ is of the α 'th Baire class.

Similarly, a map $f : X \rightarrow Y$ is called the *left B_α -compositor* (for a class \mathcal{C} of topological spaces) if for any space Z (for any $Z \in \mathcal{C}$) and a map $g \in B_\alpha(Z, X)$ the composition $f \circ g : Z \rightarrow Y$ belongs to the α 'th Baire class.

Let $\mathcal{M}_0(X)$ be the family of all functionally closed subsets of X and let $\mathcal{A}_0(X)$ be the family of all functionally open subsets of X . For every $\alpha \in [1, \omega_1)$ we put

$$\mathcal{M}_\alpha(X) = \left\{ \bigcap_{n=1}^\infty A_n : A_n \in \bigcup_{\beta < \alpha} \mathcal{A}_\beta(X), n = 1, 2, \dots \right\} \quad \text{and} \quad \mathcal{A}_\alpha(X) = \left\{ \bigcup_{n=1}^\infty A_n : A_n \in \bigcup_{\beta < \alpha} \mathcal{M}_\beta(X), n = 1, 2, \dots \right\}.$$

Elements from the class $\mathcal{M}_\alpha(X)$ belong to the α 'th functionally multiplicative class and elements from $\mathcal{A}_\alpha(X)$ belong to the α 'th functionally additive class in X . We say that a set is *functionally ambiguous* of the α 'th class if it belongs to $\mathcal{M}_\alpha(X) \cap \mathcal{A}_\alpha(X)$.

A map $f : X \rightarrow Y$ is of the α 'th (functionally) Lebesgue class, if the preimage $f^{-1}(V)$ of any open set $V \subseteq Y$ is of the α 'th (functionally) additive class α in X . The collection of all maps of the α 'th (functionally) Lebesgue class we denote by $H_\alpha(X, Y)$ ($K_\alpha(X, Y)$).

A family $\mathcal{A} = (A_i : i \in I)$ of subsets of a topological space X is called *discrete*, if every point of X has an open neighborhood which intersects at most one set from \mathcal{A} ; *strongly discrete*, if there exists a discrete family $(U_i : i \in I)$ of open subsets of X such that $\overline{A_i} \subseteq U_i$ for every $i \in I$; *strongly functionally discrete* or, briefly, *sfd family*,

if there exists a discrete family $(U_i : i \in I)$ of functionally open subsets of X such that $\overline{A_i} \subseteq U_i$ for every $i \in I$. Notice that in a metrizable space X each discrete family of sets is strongly functionally discrete.

A family \mathcal{B} of sets of a topological space X is called a *base* for a map $f : X \rightarrow Y$ if the preimage $f^{-1}(V)$ of an arbitrary open set V in Y is a union of sets from \mathcal{B} . In the case when \mathcal{B} is a countable union of (strongly functionally) discrete families, we say that f is σ -(strongly functionally) discrete and write this fact as $f \in \Sigma(X, Y)$ ($f \in \Sigma^f(X, Y)$). If a map f has a σ -(strongly functionally) discrete base which consists of (functionally) ambiguous sets of the α 'th class, then we say that f belongs to the class $\Sigma_\alpha(X, Y)$ (or to $\Sigma_\alpha^f(X, Y)$, respectively).

We will use the next result which, in fact, was established in [11] and [12].

Theorem 1. *Let $\alpha \in [0, \omega_1)$, X be a topological space, Y be a metrizable connected and locally path-connected space. Then*

- (i) $B_\alpha(X, Y) = \Sigma_\alpha^f(X, Y)$, if $\alpha < \omega_0$,
- (ii) $B_\alpha(X, Y) = \Sigma_{\alpha+1}^f(X, Y)$, if $\alpha \geq \omega_0$.

Proof. For $\alpha = 0$ the statement is evident. The equality (i) for $\alpha = 1$ was proved in [12, Theorem 5]. This fact and [11, Theorem 22] imply the equalities (i) and (ii) for all $\alpha > 1$. \square

A topological space is called *perfect* if any its open subset is F_σ .

3. AN " $\varepsilon - \delta$ " CHARACTERIZATION OF σ -DISCRETE MAPS

Lemma 2. *Let $\alpha \in [1, \omega_1)$, X be a topological space and (Y, d) be a metric space. For a map $f : X \rightarrow Y$ the following conditions are equivalent:*

- (1) $f \in \Sigma_\alpha^f(X, Y)$;
- (2) *for any $\varepsilon > 0$ there exists a σ -sfd family \mathcal{A}_ε which consists of functionally ambiguous sets of the class α in X such that $X = \bigcup \mathcal{A}_\varepsilon$ and $\text{diam} f(A) < \varepsilon$ for every $A \in \mathcal{A}_\varepsilon$.*

Proof. **(1) \Rightarrow (2).** Fix $\varepsilon > 0$. Let \mathcal{B} be a σ -sfd base for f which consists of functionally ambiguous sets of the class α in X . Consider a cover \mathcal{V} of the space Y by open balls of diameters $< \varepsilon$. Then the family

$$\mathcal{A}_\varepsilon = \{B \in \mathcal{B} : B \subseteq f^{-1}(V) \text{ for some } V \in \mathcal{V}\}$$

is required.

(2) \Rightarrow (1). For every $n \in \mathbb{N}$ we choose a σ -sfd family \mathcal{A}_n which consists of functionally ambiguous sets of the class α in X such that $X = \bigcup \mathcal{A}_n$ and $\text{diam} f(A) < \frac{1}{n}$ for every $A \in \mathcal{A}_n$. Denote $\mathcal{B} = \bigcup \mathcal{A}_n$. Then \mathcal{B} is a σ -sfd family of functionally ambiguous sets of the class α and \mathcal{B} is a base for f , since $f^{-1}(V) = \bigcup_{n=1}^{\infty} \{B \in \mathcal{B} : B \subseteq f^{-1}(V)\}$ for any open set $V \subseteq Y$. \square

By $\tau(X)$ we denote the topology of a topological space X . A multi-valued map $U : X \rightarrow \tau(X)$ is called a *neighborhood-map* if $x \in U(x)$ for every $x \in X$.

Let \mathcal{A} and \mathcal{B} be families of sets. We write $\mathcal{A} \prec \mathcal{B}$ if for every set $A \in \mathcal{A}$ there exists a set $B \in \mathcal{B}$ such that $A \subseteq B$.

Lemma 3. *Let X be a topological space and \mathcal{A} is a σ -sfd cover of X by functionally closed maps. Then*

- (1) *there exists a neighborhood-map $U : X \rightarrow \tau(X)$ such that for all $x, y \in X$*

$$(3.1) \quad x, y \in U(x) \cap U(y) \implies x, y \in A \text{ for some } A \in \mathcal{A};$$

- (2) *if X is metrizable and d_X is a metric on X which generates its topological structure, then there exists a function $\delta \in B_1(X, \mathbb{R}^+)$ such that for all $x, y \in X$*

$$(3.2) \quad d_X(x, y) < \min\{\delta(x), \delta(y)\} \implies x, y \in A \text{ for some } A \in \mathcal{A}.$$

Proof. According to [11, Lemma 13] there exists a sequence $(\mathcal{A}_n)_{n=1}^{\infty}$ of sfd families of functionally closed sets in X such that $\bigcup_{n=1}^{\infty} \mathcal{A}_n \prec \mathcal{A}$, $\bigcup_{n=1}^{\infty} \mathcal{A}_n = X$ and $\mathcal{A}_n \prec \mathcal{A}_{n+1}$ for every $n \in \mathbb{N}$.

Let $x \in X$. Denote $n(x) = \min\{n : x \in \bigcup \mathcal{A}_n\}$. Since the family $\mathcal{A}_{n(x)}$ is disjoint, there exists a unique set $A(x) \in \mathcal{A}_{n(x)}$ such that $x \in A(x)$. We put

$$U(x) = X \setminus \left(\bigcup \mathcal{A}_{n(x)} \setminus A(x) \right).$$

Notice that the set $U(x)$ is open, since the family $\mathcal{A}_{n(x)}$ is discrete.

In the case (2) for every $x \in X$ we put

$$\delta(x) = d_X(x, X \setminus U(x)).$$

We show that the neighborhood-map $U : X \rightarrow \tau(X)$ in the case (1) and the function $\delta : X \rightarrow \mathbb{R}^+$ in the case (2) satisfy the requirements of the lemma.

Let $x, y \in U(x) \cap U(y)$. Assume that $n(x) \leq n(y)$ and choose a set $B \in \mathcal{A}_{n(y)}$ such that $A(x) \subseteq B$. Since $x \in U(y)$, $x \in A(y)$. It follows that $B = A(y)$. Since $\bigcup_{n=1}^{\infty} \mathcal{A}_n \prec \mathcal{A}$, there exists a set $A \in \mathcal{A}$ with $A(y) \subseteq A$. Then $x, y \in A$.

Observe that in the case (2) for all $x, y \in X$ the inequality $d_X(x, y) < \min\{\delta(x), \delta(y)\}$ implies that $x, y \in U(x) \cap U(y)$. Hence, we get (3.2).

It remains to prove that δ is of the first Baire class. For every $n \in \mathbb{N}$ we denote

$$F_n = \bigcup \mathcal{A}_n \quad \text{and} \quad \delta_n = \delta|_{F_n}.$$

Since

$$\delta_1(x) = d_X(x, F_1 \setminus A)$$

for $x \in A$, the restriction $\delta_1|_A$ is continuous for every $A \in \mathcal{A}_1$. Then the function $\delta_1 : F_1 \rightarrow \mathbb{R}^+$ is continuous, since the family \mathcal{A}_1 is discrete and consists of closed sets. Assume that each function $\delta_1, \dots, \delta_n$ belongs to the first Baire class for some $n \geq 1$. Fix $A \in \mathcal{A}$ and notice that for all $x \in A$ we have

$$\delta_{n+1}(x) = \begin{cases} \delta_n(x), & \text{if } x \in B \text{ for some } B \in \mathcal{A}_n, \\ d_X(x, F_{n+1} \setminus A), & \text{otherwise.} \end{cases}$$

Taking into account the inductive assumption and the closedness of B , we obtain that $\delta_{n+1}|_A$ is a Baire-one function. Since the family \mathcal{A}_2 is discrete, $\delta_{n+1} \in B_1(F_{n+1}, \mathbb{R}^+)$.

Since the family $(F_n : n \in \mathbb{N})$ is a closed cover of X such that every restriction $\delta|_{F_n}$ is of the first Baire class, $\delta \in B_1(X, \mathbb{R}^+)$. \square

Lemma 4. *Let X be a topological space, (Y, d_Y) be a metric space and $f \in \Sigma_1^f(X, Y)$. Then for any $\varepsilon > 0$ there exists*

- (1) *a neighborhood-map $U : X \rightarrow \tau(X)$ such that for all $x, y \in X$*
- (3.3)
$$x, y \in U(x) \cap U(y) \implies d_Y(f(x), f(y)) < \varepsilon;$$
- (2) *a function $\delta \in B_1(X, \mathbb{R}^+)$ such that for all $x, y \in X$*
- (3.4)
$$d_X(x, y) < \min\{\delta(x), \delta(y)\} \implies d_Y(f(x), f(y)) < \varepsilon,$$

if X is metrizable and d_X is a metric on X which generates its topological structure.

Proof. Fix $\varepsilon > 0$. Applying Lemma 2 we may choose a σ -sfd family \mathcal{A} which consists of functionally closed sets such that $X = \bigcup \mathcal{A}$ and $\text{diam} f(A) < \varepsilon$ for each $A \in \mathcal{A}$. Lemma 3 implies that there exists a neighborhood-map $U : X \rightarrow \tau(X)$ such that (3.1) holds in the case (1) and there exists a function $\delta \in B_1(X, \mathbb{R}^+)$ which satisfies (3.2) (2). Therefore, if $x, y \in U(x) \cap U(y)$ in the case (1) or $d_X(x, y) < \min\{\delta(x), \delta(y)\}$ in the case (2), then there exists a set $A \in \mathcal{A}$ such that $x, y \in A$. Then $d_Y(f(x), f(y)) \leq \text{diam} f(A) < \varepsilon$. \square

Lemma 5. *Let (X, d_X) , (Y, d_Y) be metric spaces and $f : X \rightarrow Y$ be a map. If for any $\varepsilon > 0$ there exists a function $\delta : X \rightarrow \mathbb{R}^+$ such that for all $x, y \in X$ the inequality $d_Y(f(x), f(y)) < \varepsilon$ holds whenever $d_X(x, y) < \min\{\delta(x), \delta(y)\}$, then $f \in \Sigma_1^f(X, Y)$.*

Proof. We show that the condition (2) of Lemma 2 holds. Fix $\varepsilon > 0$ and take a function $\delta : X \rightarrow \mathbb{R}^+$ such that for all $x, y \in X$

$$d_X(x, y) < \min\{\delta(x), \delta(y)\} \implies d_Y(f(x), f(y)) < \frac{\varepsilon}{4}.$$

Now for every $n \in \mathbb{N}$ we choose a σ -discrete cover \mathcal{H}_n of X by closed sets of diameters $< \frac{1}{3n}$ and denote $D_n = \{x \in X : \delta(x) > \frac{1}{n}\}$. Let $\mathcal{H} = \bigcup_m \mathcal{H}_{mn}$, where each family \mathcal{H}_{mn} is discrete in X . We put

$$\mathcal{A}_{mn} = \{H \cap \overline{D_n} : H \in \mathcal{H}_{mn}\} \quad \text{and} \quad \mathcal{A} = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{A}_{mn}.$$

Then the family \mathcal{A} forms a cover of X and each family \mathcal{A}_{mn} is discrete.

Let $m, n \in \mathbb{N}$ and $x, y \in A \in \mathcal{A}_{mn}$. We choose a number $k > 3n$ with $x, y \in D_k$. Since $x, y \in \overline{D_n}$, there exist $x_1, y_1 \in D_n$ such that $d_X(x, x_1) < \frac{1}{k}$ and $d_X(y, y_1) < \frac{1}{k}$. Moreover, since $x, y \in H \in \mathcal{H}_{mn}$, we have $d_X(x, y) < \frac{1}{3n}$, which implies $d_X(x_1, y_1) < \frac{1}{n}$. Therefore,

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_1)) + d_Y(f(x_1), f(y_1)) + d_Y(f(y_1), f(y)) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}.$$

Hence, $\text{diam} f(A) \leq \frac{3\varepsilon}{4} < \varepsilon$. \square

Let $\alpha \in [0, \omega_1)$. A map $f : X \rightarrow Y$ is said to be \mathcal{M}_α -measurable, if $f^{-1}(B) \in \mathcal{M}_\alpha(X)$ for any $B \in \mathcal{M}_\alpha(Y)$.

The following fact immediately implies from the definition of \mathcal{M}_α -measurable map.

Proposition 6. Let $\alpha \in [0, \omega_1)$. For a map $f : X \rightarrow Y$ the following conditions are equivalent:

- (1) f is \mathcal{M}_α -measurable;
- (2) $f^{-1}(B) \in \mathcal{A}_\alpha(X)$ for any $B \in \mathcal{A}_\alpha(Y)$;
- (3) the preimage $f^{-1}(B)$ of any functionally ambiguous set B of the class α in Y is a functionally ambiguous set of the class α in X .

Theorem 7. Let X be a topological space, (Y, d_Y) be a metric space and $f : X \rightarrow Y$ be an \mathcal{M}_1 -measurable map from the class $\Sigma^f(X, Y)$. Then for any function $\varepsilon \in B_1(Y, \mathbb{R}^+)$ there exists a neighborhood-map $U : X \rightarrow \tau(X)$ such that for all $x, y \in X$

$$x, y \in U(x) \cap U(y) \implies d_Y(f(x), f(y)) < \min\{\varepsilon(f(x)), \varepsilon(f(y))\}.$$

Proof. Fix any Baire-one function $\varepsilon : Y \rightarrow \mathbb{R}^+$. Notice that for every $n \in \mathbb{N}$ the set

$$A_n = f^{-1}(\varepsilon^{-1}(\frac{1}{2^n}, +\infty))$$

belongs to the class $\mathcal{A}_1(X)$. By the Reduction Theorem for sets of functional classes [9, Lemma 3.1] (see also [16, p. 350]) there exists a partition $(B_n : n \in \mathbb{N})$ of X by functionally ambiguous sets such that $B_n \subseteq A_n$ for every n . Lemma 3 implies that there exists a neighborhood-map $V : X \rightarrow \tau(X)$ such that for all $x, y \in X$

$$x, y \in V(x) \cap V(y) \implies x, y \in B_n \text{ for some } n \in \mathbb{N}.$$

Since $f \in \Sigma^f(X, Y) \cap K_1(X, Y)$, it follows from Lemma 4 that for every $n \in \mathbb{N}$ there exists a neighborhood-map $V_n : X \rightarrow \tau(X)$ such that for all $x, y \in X$

$$x, y \in V_n(x) \cap V_n(y) \implies d_Y(f(x), f(y)) < \frac{1}{2^n}.$$

For every $x \in X$ we put

$$U(x) = V(x) \cap V_n(x),$$

if $x \in B_n$ for some $n \in \mathbb{N}$. It is easy to see that $U : X \rightarrow \tau(X)$ is required neighborhood-map. \square

4. BARELY CONTINUOUS σ -DISCRETE MAPS

We start this section with a generalization of the classical theorem of R. Baire on a classification of pointwise discontinuous maps.

Recall that a map $f : X \rightarrow Y$ is *barely continuous*, if the restriction $f|_F$ on any non-empty close set $F \subseteq X$ has a point of continuity. A map f is *pointwise discontinuous*, if the set $C(f)$ of all continuity points of f is dense in X .

Notice that if X is a hereditarily Baire space, then f is barely continuous if and only if for any non-empty closed set $F \subseteq X$ the restriction $f|_F$ is pointwise discontinuous.

A topological space X is called *contractible*, if there exist a point $x_0 \in X$ and a continuous map $\gamma : X \times [0, 1] \rightarrow X$ such that $\gamma(x, 0) = x$ and $\gamma(x, 1) = x_0$ for all $x \in X$.

Lemma 8. Let $\alpha \in [1, \omega_1)$, X be a topological space, $G \subseteq X$ be a functionally open set, Y be a contractible space, $y_0 \in Y$ and $g \in B_1(G, Y)$. Then the formula

$$f(x) = \begin{cases} g(x), & \text{if } x \in G, \\ y_0, & \text{if } x \in X \setminus G \end{cases}$$

defines an extension $f \in B_1(X, Y)$ of the map g .

Proof. Let $\gamma : Y \times [0, 1] \rightarrow Y$ be a continuous map such that $\gamma(y, 0) = y$ and $\gamma(y, 1) = y_0$ for all $y \in Y$.

Consider a continuous function $\varphi : X \rightarrow [0, 1]$ with $G = \varphi^{-1}((0, 1])$. For every $n \in \mathbb{N}$ we put

$$F_{n,1} = X \setminus G, \quad F_{n,2} = \varphi^{-1}([\frac{1}{n+1}, 1]),$$

$$U_{n,1} = \varphi^{-1}([0, \frac{1}{n+3})), \quad U_{n,2} = \varphi^{-1}((\frac{1}{n+2}, 1]).$$

Moreover, for all $n \in \mathbb{N}$ and $i \in \{1, 2\}$ we choose a continuous function $\psi_{n,i} : X \rightarrow [0, 1]$ such that

$$U_{n,i} = \psi_{n,i}^{-1}((0, 1]) \text{ and } F_{n,i} = \psi_{n,i}^{-1}(1).$$

Take a sequence $(g_n)_{n=1}^\infty$ of continuous maps $g_n : G \rightarrow Y$ which is pointwise convergent to g on G . For every $n \in \mathbb{N}$ let $g_{n,1}(x) = y_0$ for all $x \in X$ and $g_{n,2}(x) = g_n(x)$ for all $x \in G$. We define a map $f_n : X \rightarrow Y$ as follows:

$$f_n(x) = \begin{cases} \gamma(g_{n,i}(x), 1 - \psi_{n,i}(x)), & \text{if } x \in U_{n,i} \text{ for some } i \in \{1, 2\}, \\ y_0, & \text{otherwise.} \end{cases}$$

Fix $x \in X$. If $x \in X \setminus G$, then $f_n(x) = \gamma(g_{n,1}(x), 0) = g_{n,1}(x) = y_0 = f(x)$ for all $n \in \mathbb{N}$. If $x \in G$, then there exists n_0 such that $f_n(x) = \gamma(g_{n,2}(x), 0) = g_{n,2}(x)$ for all $n \geq n_0$. Hence, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

It is easy to see that each map $f_n : X \rightarrow Y$ is continuous. Therefore, $f \in B_1(X, Y)$. \square

Let X be a topological space, (Y, d_Y) be a metric space and $\varepsilon > 0$. We say that a map $f : X \rightarrow Y$ can be ε -approximated by a Baire-one map on X , if there exists a map $g \in B_1(X, Y)$ such that $d_Y(f(x), g(x)) \leq \varepsilon$ for all $x \in X$; locally ε -approximated by Baire-one maps on X , if for every $x \in X$ there is a neighborhood U_x of x such that the restriction $f|_{U_x}$ can be ε -approximated by a Baire-one map on U_x .

Notice that analogs of Lemma 9 and Theorem 10 for $Y = \mathbb{R}$ were proved in [21].

Lemma 9. *Let X be a Hausdorff paracompact space, (Y, d_Y) be a metric contractible locally path-connected space, $\varepsilon > 0$ and let a map $f : X \rightarrow Y$ can be locally ε -approximated by Baire-one maps. Then f can be ε -approximated by a Baire-one map on X .*

Proof. Since X is paracompact, there exists a σ -discrete cover \mathcal{U} of X by functionally open sets such that for each $U \in \mathcal{U}$ we may choose a map $f_U \in B_1(U, Y)$ with $d_Y(f(x), f_U(x)) \leq \varepsilon$ for all $x \in U$. Let $\mathcal{U} = \bigcup_n \mathcal{U}_n$, where \mathcal{U}_n is a discrete family of functionally open sets in X . Denote $U_n = \bigcup \mathcal{U}_n$ for every $n \in \mathbb{N}$ and notice that each set U_n is functionally open. Therefore, by Lemma 8 for every $n \in \mathbb{N}$ the map $f_n \in B_1(U_n, Y)$, defined by the equality

$$f_n(x) = f_U(x), \text{ if } x \in U \in \mathcal{U}_n,$$

can be extended to a map $g_n \in B_1(X, Y)$.

We put

$$\mathcal{A}_1 = \mathcal{U}_1 \text{ and } \mathcal{A}_{n+1} = (U \setminus \bigcup_{k=1}^n \mathcal{U}_k : U \in \mathcal{U}_{n+1}) \text{ for } n \geq 1.$$

Then each set $A_n = \bigcup \mathcal{A}_n$ is functionally ambiguous in X and the family $(A_n : n \in \mathbb{N})$ forms a partition of X . Applying [11, Proposition 8] we have that the formula

$$g(x) = g_n(x), \text{ if } x \in A_n \text{ for some } n \in \mathbb{N}$$

defines the map $g \in \Sigma_1^f(X, Y)$. Hence, $g \in B_1(X, Y)$ by Theorem 1. Moreover, for every $x \in X$ we have

$$d_Y(f(x), g(x)) \leq \varepsilon,$$

since g is an extension of g_n , which is an extension of f_n . \square

For a point x of a topological space X the system of all neighborhoods of x is denoted by \mathcal{U}_x . For a map $f : X \rightarrow Y$ between a topological space X and a metric space (Y, d_Y) , and for a set $A \subseteq X$ we put

$$\omega_f(A) = \sup_{x, y \in A} d_Y(f(x), f(y)), \quad \omega_f(x) = \inf_{U \in \mathcal{U}_x} \omega_f(U),$$

We say that $f : X \rightarrow (Y, d_Y)$ is *weakly barely continuous*, if for every $\varepsilon > 0$ and for every non-empty closed set $F \subseteq X$ there exists a point $x \in F$ such that $\omega_{f|_F}(x) < \varepsilon$. Evidently, each barely continuous map is weakly barely continuous. The converse it not true, as a restriction of the Riemann function shows. Indeed, let $f(x) = \frac{1}{n}$ if $x = r_n \in \mathbb{Q}$ for some $n \in \mathbb{N}$ and $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $f|_{\mathbb{Q}}$ is weakly barely continuous, but is not a barely continuous function.

Theorem 10. *Let X be a perfect paracompact space, (Y, d_Y) be a metric contractible locally path-connected space and $f : X \rightarrow Y$ be a weakly barely continuous map. Then $f \in B_1(X, Y)$.*

Proof. Fix $\varepsilon > 0$ and show that f can be locally ε -approximated by Baire-one maps. Let \mathcal{G} be a collection of all open subsets of X such that f is ε -approximated by Baire-one maps on each set from \mathcal{G} . We put $G = \bigcup \mathcal{G}$ and notice that G is open and non-empty, since it contains all continuity points of f . Moreover, since G is F_σ in X , G is a paracompact functionally open subset of X . Applying Lemma 9 we obtain the existence of a map $h \in B_1(G, Y)$ such that

$$d_Y(h(x), f(x)) \leq \varepsilon$$

for all $x \in G$.

We put $F = X \setminus G$ and prove that $F = \emptyset$. Assume the contrary. Since f is weakly barely continuous, we may choose a point $x_0 \in F$ and an open neighborhood U_0 of x_0 such that

$$d_Y(f(x), f(x_0)) < \varepsilon$$

for all $x \in U_0 \cap F$. Let $y_0 = f(x_0)$. By Lemma 8 the map $g : X \rightarrow Y$, defined as follows

$$g(x) = \begin{cases} h(x), & x \in G, \\ y_0, & x \in F, \end{cases}$$

is of the first Baire class. Notice that

$$d_Y(f(x), g(x)) \leq \varepsilon$$

for all $x \in U_0$. Therefore, $U_0 \subseteq G$. In particular, $x_0 \in G$, which implies a contradiction. Hence, $G = X$.

Lemma 9 implies that there exists a sequence $(f_n)_{n=1}^\infty$ of Baire-one maps with that

$$d_Y(f(x), f_n(x)) \leq \frac{1}{n}.$$

Since a uniform limit of a sequence of Baire-one maps with a metrizable connected and locally path-connected range space remains a Baire-one map [13, Theorem 2], $f \in B_1(X, Y)$. \square

Theorem 11. *Let X be a perfect paracompact space, Y be a metrizable space and $f : X \rightarrow Y$ be a weakly barely continuous map. Then $f \in \Sigma_1(X, Y)$.*

Proof. Let $Z = \ell_\infty(Y)$, $\varphi : Y \rightarrow Z$ is a homeomorphic embedding and $h = \varphi \circ f$. By Theorem 10, $h \in B_1(X, Z)$. Then it follows from Theorem 1 that h together with f have a σ -discrete base \mathcal{B} which consists of ambiguous sets in X . \square

Remark 1. *Notice that if X is not perfect, Theorem 11 is not valid. Indeed, let D be an uncountable discrete space and $X = \alpha D$ be its one-point compactification. Consider a bijection $f : X \rightarrow D$. It is easy to see that f is barely continuous. Assume that f has a base $\mathcal{B} = \bigcup_n \mathcal{B}_n$ in X , where every family \mathcal{B}_n is discrete in X . The compactness of X implies that every family \mathcal{B}_n is finite. On the other hand, all singletons are elements of \mathcal{B} , which implies a contradiction.*

Theorem 12. *Let X be a hereditarily Baire metrizable space and Y be a metrizable space. For a map $f : X \rightarrow Y$ the following conditions are equivalent:*

- (1) $f \in \Sigma_1(X, Y)$,
- (2) $f \in H_1(X, Y)$,
- (3) f is barely continuous,
- (4) for any $\varepsilon > 0$ there exists a function $\delta \in B_1(X, \mathbb{R}^+)$ such that (1.3) for all $x, y \in X$, where d_X is a metric on X which generates its topology;
- (5) for any $\varepsilon > 0$ there exists a function $\delta : X \rightarrow \mathbb{R}^+$ such that (1.3) holds for all $x, y \in X$, where d_X is a metric on X which generates its topology.

Proof. The implication (1) \Rightarrow (2) follows from [7, Proposition 2]. The implication (2) \Rightarrow (3) was proved by G. Koumoullis [15, Theorem 4.12]. Theorem 11 implies (3) \Rightarrow (1). Lemma 4 implies (1) \Rightarrow (4). Evidently, (4) \Rightarrow (5). Finally, the implication (5) \Rightarrow (1) is proved in Lemma 5. \square

Remark 2. *The implication (3) \Rightarrow (2) for a perfect paracompact space X and a perfect space Y was established in [14, Theorem 6]. The implication (5) \Rightarrow (3) was also proved in [14, Theorem 5].*

5. A CHARACTERIZATION OF RIGHT COMPOSITORS

The class of all right B_α -compositors between X and Y we will denote by $RB_\alpha(X, Y)$.

Proposition 13. *Let X, Y be topological spaces and $\alpha \in [1, \omega_1)$. Then*

$$\overline{\bigcup_{\beta < \alpha} RB_\beta(X, Y)}^{\text{st}} \subseteq RB_\alpha(X, Y).$$

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence of right B_{β_n} -compositors which is convergent pointwisely to $f : X \rightarrow Y$ and $\beta_n < \alpha$ for every $n \in \mathbb{N}$. Consider a topological space Z and a map $g \in B_\alpha(Y, Z)$. Choose a sequence $(g_n)_{n=1}^\infty$ of maps $g_n \in B_{\gamma_n}(Y, Z)$ which converges pointwisely to g and $\gamma_n < \alpha$ for every $n \in \mathbb{N}$. Denote $\delta_n = \max\{\beta_n, \gamma_n\}$ and observe that each map f_n is the right B_{δ_n} -compositor. Then $h_n = g_n \circ f_n \in B_{\delta_n}(X, Z)$. Moreover, if $x \in X$, then $f_n(x) = f(x)$ for all $n \geq N$, which implies that $\lim_{n \rightarrow \infty} h_n(x) = \lim_{n \rightarrow \infty} g_n(f(x)) = g(f(x))$. Hence, $f \in RB_\alpha(X, Y)$. \square

The following result was proved in [6, 3.2 (4), p.160] for metrizable spaces X, Y, Z .

Lemma 14. *Let X, Y, Z be topological spaces, $f \in \Sigma^f(X, Y)$, $g \in \Sigma^f(Y, Z)$ and $h = g \circ f$. Then $h \in \Sigma^f(X, Z)$.*

Proof. Let $\mathcal{A} = \bigcup_n \mathcal{A}_n$ be a σ -sdf base for f , $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a σ -sdf base for g and each of the families \mathcal{A}_n and \mathcal{B}_n is strongly functionally discrete in X and Y , respectively. For every n we choose a discrete family $\mathcal{U}_n = (U_B : B \in \mathcal{B}_n)$ of functionally open sets in Y such that $B \subseteq U_B$ for every $B \in \mathcal{B}_n$. For all $n, m \in \mathbb{N}$ we put

$$\mathcal{C}_{nm} = \{A \cap f^{-1}(B) : A \in \mathcal{A}_n, B \in \mathcal{B}_m \text{ and } f(A) \subseteq U_B\}.$$

Then the family \mathcal{C}_{nm} is strongly functionally discrete, since each \mathcal{A}_n is strongly functionally discrete and for every $A \in \mathcal{A}_n$ there exists at most one $B \in \mathcal{B}_m$ with $f(A) \subseteq U_B$.

It remains to show that the family

$$\mathcal{C} = \bigcup_n \bigcup_m \mathcal{C}_{nm}$$

is a base for h . Let W be an open set in Z . Since \mathcal{B} is a base for g , $g^{-1}(W) = \bigcup \mathcal{B}_W$, where $\mathcal{B}_W \subseteq \mathcal{B}$. Then

$$(5.1) \quad h^{-1}(W) = \bigcup \{f^{-1}(B) : B \in \mathcal{B}_W\}.$$

For each $B \in \mathcal{B}_W$ we take a family $\mathcal{A}_B \subseteq \mathcal{A}$ such that $f^{-1}(U_B) = \bigcup \mathcal{A}_B$. Then

$$f^{-1}(B) = \bigcup \{A \cap f^{-1}(B) : A \in \mathcal{A}_B, f(A) \subseteq U_B\}$$

for every $B \in \mathcal{B}_W$. Taking into account (5.1), we obtain that

$$h^{-1}(W) = \bigcup \{A \cap f^{-1}(B) : B \in \mathcal{B}_W, A \in \mathcal{A}_B, f(A) \subseteq U_B\}.$$

Therefore, \mathcal{C} is a base for h . □

Let $\alpha \in [0, \omega_1)$. A map $f : X \rightarrow Y$ is called \mathcal{M}_α -measurable, if the preimage $f^{-1}(B)$ of any set $B \in \mathcal{M}_\alpha(Y)$ belongs to the class $\mathcal{M}_\alpha(X)$.

Remark 3. Evidently, a map is \mathcal{M}_α -measurable if the preimage of any set from $\mathcal{A}_\alpha(Y)$ belongs to $\mathcal{A}_\alpha(X)$. Since each set of the α 'th functionally additive class is a union of a sequence of functionally ambiguous sets of the class α , a map is \mathcal{M}_α -measurable if and only if the preimage of any functionally ambiguous set of the class α in Y is functionally ambiguous of the class α in X .

Theorem 15. Let X be a topological space, Y be a metrizable space, $\alpha \in [1, \omega_1)$ and \mathcal{Z} be the class of all metrizable connected and locally path-connected spaces. Let $\beta = \alpha$ if $\alpha < \omega_0$ and $\beta = \alpha + 1$ if $\alpha \geq \omega_0$. For a map $f : X \rightarrow Y$ the following conditions are equivalent:

- (1) f is the right B_α -compositor for the class \mathcal{Z} ;
- (2) f is \mathcal{M}_β -measurable and σ -strongly functionally discrete.

Proof. **(1) \Rightarrow (2).** Keeping in mind Remark 3, we consider a functionally ambiguous set B of the class β in Y and its characteristic function $g : Y \rightarrow \mathbb{R}$. Notice that $g \in H_\beta(Y, \mathbb{R})$, which implies that $g \in B_\alpha(Y, \mathbb{R})$ by [16, p. 393]. Then $g \circ f \in B_\alpha(X, \mathbb{R}) = \Sigma_\beta^f(X, \mathbb{R})$. Since

$$f^{-1}(B) = (g \circ f)^{-1}((0, 2)) = (g \circ f)^{-1}(\{1\}),$$

the set $f^{-1}(B)$ is functionally ambiguous of the class β in X . Therefore, f is \mathcal{M}_β -measurable.

Now we consider the space $Z = \ell_\infty(Y) \in \mathcal{Z}$ and a homeomorphic embedding $\varphi : Y \rightarrow Z$. Condition (1) implies that the composition $\tilde{f} = \varphi \circ f : X \rightarrow Z$ belongs to the class $B_\alpha(X, Z)$. By [11, Corollary 19], $\tilde{f} \in \Sigma^f(X, Z)$. It is easy to see that a σ -sfd base \mathcal{B} for \tilde{f} is also a base for f . Hence, $f \in \Sigma^f(X, Y)$.

(2) \Rightarrow (1). Let $Z \in \mathcal{Z}$ and $g \in B_\alpha(Y, Z)$. Denote $h = g \circ f$. We show that $h \in K_\beta(X, Z)$. Let F be a closed set in Z . Then $g^{-1}(F) \in \mathcal{M}_\beta(Y)$, which implies that $h^{-1}(F) = f^{-1}(g^{-1}(F)) \in \mathcal{M}_\beta(X)$, because f is \mathcal{M}_β -measurable.

Notice that $g \in \Sigma^f(Y, Z)$. It follows from Lemma 14 that h is a σ -sfd map.

Hence, $h \in \Sigma_\beta^f(X, Z)$ according to [11, Theorem 6]. Theorem 1 implies that $h \in B_\alpha(X, Z)$. □

Remark 4. If $\alpha = 0$, then conditions (1) and (2) are equivalent to the continuity of f .

A map $f : X \rightarrow Y$ between topological spaces is *functionally piecewise continuous* if there exists a cover $(F_n : n \in \mathbb{N})$ of X by functionally closed sets such that each restriction $f|_{F_n}$ is continuous.

For a topological space X and a pair (Y, B) of a topological space Y and its subspace B we shall write $(Y, B) \in \text{AE}^f(X)$ if each continuous map $f : F \rightarrow B$ defined on a functionally closed set $F \subseteq X$ can be extended to a continuous map $g : X \rightarrow Y$. We define a space Y to belong to the class $\sigma\text{AE}^f(X)$ if there exists a countable cover $(Y_n : n \in \mathbb{N})$ of Y by closed G_δ -sets such that $(Y, Y_n) \in \text{AE}^f(X)$ for all $n \in \mathbb{N}$.

Theorem 16. Let X be a topological space, Y be a metrizable space. For a map $f : X \rightarrow Y$ the following conditions are equivalent:

- (1) $f \in B_1^{\text{st}}(X, Y)$;
 - (2) f is functionally piecewise continuous;
 - (3) for any function $\varepsilon : Y \rightarrow \mathbb{R}^+$ there exists a neighborhood-map $U : X \rightarrow \tau(X)$ such that
- $$(5.2) \quad x, y \in U(x) \cap U(y) \implies d_Y(f(x), f(y)) < \min\{\varepsilon(f(x)), \varepsilon(f(y))\}$$

for all $x, y \in X$;

- (4) for any function $\varepsilon \in B_1(Y, \mathbb{R}^+)$ here exists a neighborhood-map $U : X \rightarrow \tau(X)$ such that (5.2) holds for all $x, y \in X$;
- (5) f is the right B_1 -compositor for the class \mathcal{Z} of all metrizable connected and locally path-connected spaces;
- (6) f is \mathcal{M}_1 -measurable and σ -strongly functionally discrete.

Then (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) and (5) \Leftrightarrow (6).

If X is a metrizable space, then (4) \Rightarrow (5).

If X is a hereditarily Baire perfect paracompact space Preiss-Simon space, then (6) \Rightarrow (2).

If Y is a path-connected space and $Y \in \sigma\text{AE}^f(X)$, then (2) \Rightarrow (1).

Proof. (1) \Rightarrow (2). Denote $\Delta = \{(y, y) : y \in Y\}$ and let $(f_n)_{n=1}^\infty$ be a sequence of continuous maps $f_n : X \rightarrow Y$ which converges stably to f . For $k, n \in \mathbb{N}$ we put

$$X_{k,n} = \{x \in X : f_k(x) = f_n(x)\} \quad \text{and} \quad X_n = \bigcap_{k=n}^\infty X_{k,n}.$$

Clearly, $X_n \subseteq X_{n+1}$, $X = \bigcup_{n=1}^\infty X_n$ and $f_n|_{X_n} = f|_{X_n}$ for every $n \in \mathbb{N}$.

For all $x \in X$ and $k, n \in \mathbb{N}$ we consider a continuous map $h_{k,n} : X \rightarrow Y^2$, $h_{k,n}(x) = (f_k(x), f_n(x))$. Since $X_{k,n} = h_{k,n}^{-1}(\Delta)$ and the set Δ is functionally closed in Y^2 , $X_{k,n}$ is functionally closed in X .

(2) \Rightarrow (3). Let $(F_n)_{n=1}^\infty$ be a sequence of functionally closed sets which covers X and the restriction $f|_{F_n}$ is continuous for every n . We put $A_1 = F_1$ and $A_n = F_n \setminus (F_1 \cup \dots \cup F_{n-1})$ for $n > 1$. Fix $\varepsilon : Y \rightarrow \mathbb{R}^+$. Lemma 3 allows to take a neighborhood-map $V : X \rightarrow \tau(X)$ satisfying condition (3.1) for some σ -sfd cover $\mathcal{A} \prec (A_n : n \in \mathbb{N})$ of the space X by functionally closed sets. For every $n \in \mathbb{N}$ we use the continuity of the restriction $f_n|_{A_n}$ and choose a neighborhood-map $U_n : A_n \rightarrow \tau(A_n)$ such that for all $x, y \in X$ the inclusion $x, y \in U_n(x) \cap U_n(y)$ implies the inequality $d_Y(f_n(x), f_n(y)) < \min\{\varepsilon(f_n(x)), \varepsilon(f_n(y))\}$. For every n we take a map $V_n : A_n \rightarrow \tau(X)$ such that $V_n(x) \cap A_n = U_n(x)$ for all $x \in A_n$. Now for all $x \in X$ we put

$$U(x) = V(x) \cap V_n(x),$$

if $x \in A_n$ for some n . It is easy to see that the neighborhood-map $U : X \rightarrow \tau(X)$ satisfies the required properties.

The implication (3) \Rightarrow (4) is evident.

Assume that X is a metrizable space and prove (4) \Rightarrow (5). Let $Z \in \mathcal{Z}$, d_X and d_Z be metrics on X and Z , which generate topological structures of these spaces and let $g \in B_1(Y, Z)$. We show that the composition $h = g \circ f$ belongs to $\Sigma_1^f(X, Z)$. Fix $\varepsilon > 0$. Applying Theorem 1, we get $g \in \Sigma_1^f(Y, Z)$. Then Lemma 4 (b) implies that there exists a function $\gamma \in B_1(Y, \mathbb{R}^+)$ such that for all $x, y \in Y$ we have

$$d_Y(x, y) < \min\{\gamma(x), \gamma(y)\} \implies d_Z(g(x), g(y)) < \varepsilon.$$

According to condition (4) there exists a neighborhood-map $U : X \rightarrow \tau(X)$ such that $x, y \in X$

$$x, y \in U(x) \cap U(y) \implies d_Y(f(x), f(y)) < \min\{\gamma(f(x)), \gamma(f(y))\}.$$

For all $x \in X$ we put

$$\delta(x) = d_X(x, X \setminus U(x))$$

and obtain a continuous function $\delta : X \rightarrow \mathbb{R}^+$ which satisfies the condition of Lemma 5 for the function $h : X \rightarrow Z$. Hence, $h \in \Sigma_1^f(X, Z)$. Again by Theorem 1 we get $h \in B_1(X, Z)$.

The equivalence (5) \Leftrightarrow (6) was proved in Theorem 15.

The implication (6) \Rightarrow (2) follows from [1, Theorem 8.1].

The implication (2) \Rightarrow (1) can be proved completely similarly to the proof of Theorem 6.3 from [1], which shows this implication for a normal space X and a path-connected $Y \in \sigma\text{AE}^f(X)$. \square

Remark 5. According to Proposition 13, a stable limit of right B_1 -compositors for the class of metrizable connected and locally path-connected spaces is the right B_2 -compositor. Then each function $f \in B_2^{\text{st}}(\mathbb{R}, \mathbb{R})$ is the right B_2 -compositor. We observe that the class of all right B_2 -compositors is strictly wider than the class $B_2^{\text{st}}(\mathbb{R}, \mathbb{R})$. Indeed, consider the increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \sum_{r_n \leq x} \frac{1}{2^n},$$

where $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$. Since $f \in B_1(\mathbb{R}, \mathbb{R})$, f is \mathcal{M}_2 -measurable. By Theorem 15, f is the right B_2 -compositor. But the discontinuity points set of f is \mathbb{Q} . Therefore, $f \notin \bigcup_{\alpha < \omega_1} B_\alpha^{\text{st}}(\mathbb{R}, \mathbb{R})$.

6. LEFT COMPOSITORS

Let

- $H_\alpha^0(X, Y)$ be a set of all maps from $H_\alpha(X, Y)$ with a finite range;
- $H_\alpha^{\mathcal{M}}(X, Y)$ be a set of all maps from $H_\alpha(X, Y)$ with a metrizable separable range.

The next result follows from [16, pp. 389–391].

Theorem 17. Let X be a perfectly normal space, Y be a topological space and $\alpha \in [1, \omega_1)$. Then

- (i) $H_{\alpha+1}^M(X, Y) \subseteq \overline{H_\alpha^0(X, Y)}^P$, if α is isolated;
- (ii) $H_{\alpha+1}^M(X, Y) \subseteq \overline{H_{<\alpha}^0(X, Y)}^P$, if α is limit.

We call a topological space X *densely connected*, if for any open non-empty sets U_1, \dots, U_n in X there exists a continuous map $\gamma : [1, n] \rightarrow X$ such that $\gamma(i) \in U_i$ for every $i \in \{1, \dots, n\}$.

Notice that each space with dense path-connected subspace is densely connected.

Lemma 18. *Let X be a perfectly normal space, Y be a densely connected first countable T_1 -space and $\alpha \in [1, \omega_1)$. Then*

- (1) $H_\alpha^0(X, Y) \subseteq B_\alpha(X, Y)$, if $\alpha < \omega_0$;
- (2) $H_{\alpha+1}^0(X, Y) \subseteq B_\alpha(X, Y)$, if $\alpha \geq \omega_0$.

Proof. (1). Assume $\alpha < \omega_0$, $f \in H_\alpha^0(X, Y)$, $f(X) = \{y_1, \dots, y_n\}$, where $y_i \neq y_j$ for $i \neq j$, and $A_i = f^{-1}(y_i)$, $i = 1, \dots, n$. Then $X = \bigcup_{i=1}^n A_i$ and each set A_i is ambiguous of the class α . For every $i = 1, \dots, n$ we take an increasing sequence $(A_{i,k})_{k=1}^\infty$ of sets of multiplicative classes $< \alpha$ and a decreasing sequence $(V_{i,k})_{k=1}^\infty$ of open neighborhoods of y_i such that

$$A_i = \bigcup_{k=1}^\infty A_{i,k} \quad \text{and} \quad \{y_i\} = \bigcap_{k=1}^\infty V_{i,k}.$$

Since for every $k \in \mathbb{N}$ the family $(A_{i,k} : i = 1, \dots, n)$ is disjoint, it follows from [10, Lemma 2.1] that there exists a function $g_k \in B_{<\alpha}(X, [1, n])$ such that

$$A_{i,k} = g_k^{-1}(i)$$

for all $i = 1, \dots, n$. Moreover, since Y is densely connected, for every $k \in \mathbb{N}$ we choose a continuous map $\gamma_k : [1, n] \rightarrow Y$ such that

$$\gamma_k(i) \in V_{i,k}$$

for $i = 1, \dots, n$. We put

$$f_k = \gamma_k \circ g_k$$

for every $k \in \mathbb{N}$ and obtain a sequence $(f_k)_{k=1}^\infty$ of maps $f_k \in B_{<\alpha}(X, Y)$ which is pointwise convergent to f on X . Hence, $f \in B_\alpha(X, Y)$.

(2). We argue by the induction on α . Let $\alpha = \omega_0$ and $f \in H_{\alpha+1}^0(X, Y)$. By Theorem 17 there exists a sequence of maps $f_n \in H_n^0(X, Y)$ which is pointwise convergent to f on X . It follows from the previous arguments that $f_n \in B_n(X, Y)$. Therefore, $f \in B_{\omega_0}^0(X, Y)$.

Now we suppose that (2) is true for all $\beta \in [\omega_0, \alpha)$, where $\alpha < \omega_1$, and consider a map $f \in H_{\alpha+1}^0(X, Y)$.

Let α be a limit ordinal. Applying Theorem 17 we choose a sequence of maps $(f_n)_{n=1}^\infty$ which converges pointwise to f on X and $f_n \in H_{<\alpha}^0(X, Y)$. By the inductive assumption we have $f_n \in B_{<\alpha}(X, Y)$ which implies that $f \in B_\alpha(X, Y)$.

If α is isolated, then $\alpha = \lambda + m$, where λ is limit and $m \in \mathbb{N}$. By the induction on m we obtain that $H_{\lambda+m+1}^0(X, Y) \subseteq B_{\lambda+m}(X, Y)$. \square

Theorem 17 and Lemma 18 imply

Theorem 19. *Let X be a perfectly normal space, Y be a first countable densely connected T_1 -space and $\alpha \in [1, \omega_1)$. Then*

- (1) $H_{\alpha+1}^M(X, Y) \subseteq B_{\alpha+1}(X, Y)$, if $\alpha < \omega_0$;
- (2) $H_{\alpha+1}^M(X, Y) \subseteq B_\alpha(X, Y)$, if $\alpha \geq \omega_0$.

Theorem 20. *Let X be a T_1 -space, Y be a perfectly normal space, $f : X \rightarrow Y$ be a map and $\alpha \in [1, \omega_1)$. If*

- (a) X is a connected and locally path-connected metrizable space and $\alpha = 1$, or
- (b) X is a first countable densely connected space and $\alpha > 1$,

then the following conditions are equivalent:

- (1) f is continuous;
- (2) f is the left B_α -compositor.

Proof. Since the implication (1) \Rightarrow (2) is obvious, we prove (2) \Rightarrow (1). Assume that f is discontinuous at $x_0 \in X$ and choose a sequence $(x_n)_{n=1}^\infty$ of points from X and a neighborhood V of $y_0 = f(x_0)$ in Y such that $\lim_{n \rightarrow \infty} x_n = x_0$ and $y_n = f(x_n) \in Y \setminus V$ for every $n \in \mathbb{N}$.

We take a set $A \subseteq \mathbb{R}$ such that $A \in \mathcal{A}_\beta \setminus \mathcal{M}_\beta$, where $\beta = \alpha$ if $\alpha < \omega_0$, and $\beta = \alpha + 1$ if $\alpha \geq \omega_0$. Let $(A_n)_{n=1}^\infty$ be a sequence of sets such that $A = \bigcup_{n=1}^\infty A_n$ and $A_n \in \mathcal{M}_{<\beta}$. We set

$$B_1 = A_1 \quad \text{and} \quad B_n = A_n \setminus (A_1 \cup \dots \cup A_{n-1}) \quad \text{for } n \geq 2.$$

Then $(B_n)_{n=1}^\infty$ is a disjoint sequence of ambiguous sets of the class β . Moreover, $A = \bigcup_{n=1}^\infty B_n$.

We define a map $g : \mathbb{R} \rightarrow X$ as follows:

$$(6.1) \quad g(t) = \begin{cases} x_n, & t \in B_n \text{ for some } n \in \mathbb{N}, \\ x_0, & t \in \mathbb{R} \setminus A \end{cases}$$

and show that $g \in B_\alpha(\mathbb{R}, X)$. Since the sequence $(x_n)_{n=1}^\infty$ converges to x_0 , $g \in H_\beta(\mathbb{R}, X)$. In the case (a) it follows from [5, Theorem 1] that $g \in B_1(\mathbb{R}, X)$. In the case (b) we observe that the space $X_0 = \{x_n : n = 0, 1, \dots\}$ is metrizable and separable. Then we have that $g \in B_\alpha(\mathbb{R}, X)$ by Theorem 19.

According to the assumption, the composition $h = f \circ g : \mathbb{R} \rightarrow Y$ belongs to the class $B_\alpha(\mathbb{R}, Y)$. On the other hand,

$$h(t) = \begin{cases} y_n, & t \in B_n \text{ for some } n \in \mathbb{N}, \\ y_0, & t \in \mathbb{R} \setminus A. \end{cases}$$

Then $h^{-1}(V) = \mathbb{R} \setminus A$. Hence, $h \in H_\beta(\mathbb{R}, Y)$ by Theorem 1 from [16, p. 386]. This implies a contradiction, since $\mathbb{R} \setminus A \notin \mathcal{A}_\beta$. \square

REFERENCES

- [1] T. Banach, B. Bokalo *On scatteredly continuous maps between topological spaces*, Topology Appl. **157** (1) (2010), 108–122.
- [2] J. P. Fenecios, E. A. Cabral, *k*-continuous functions and right B_1 -compositors, J. Indones. Math. Soc. Vol. 18, No. 1 (2012), 37–44.
- [3] J. P. Fenecios, E. A. Cabral, *Left Baire-one compositors and continuous functions*, Int. J. of Math. and Math. Sci. (2013).
- [4] J. P. Fenecios, E. A. Cabral, *A simpler proof for the $\varepsilon - \delta$ characterization of Baire class one functions*, Real Anal. Exch. **39** (2), (2013/2014), 441–446.
- [5] M. Fosgerau, *When are Borel functions Baire functions?*, Fund. Math. **143** (1993), 137–152.
- [6] R.W. Hansell, *Borel measurable mappings for nonseparable metric spaces*, Trans. Amer. Math. Soc. 161 (1971), 145–169.
- [7] R.W. Hansell, *On Borel mappings and Baire functions*, Trans. Amer. Math. Soc. 194 (1974), 195–211.
- [8] J. E. Jayne, C. A. Rogers, *First level Borel functions and isomorphisms*, J. Math. Pures Appl. 61 (1982), 177–205.
- [9] O. Karlova, *On α -embedded sets and extension of mappings*, Comment. Math. Univ. Carolin. **54**:3 (2013), 377–396.
- [10] O. Karlova, *Classification of separately continuous functions with values in σ -metrizable spaces*, Appl. Gen. Topol., **13** (2) (2012) 167–178.
- [11] O. Karlova, *Functionally σ -discrete mappings and a generalization of Banach's theorem*, Top. Appl. 189 (2015), 92–106.
- [12] O. Karlova, *On Baire classification of mappings with values in connected spaces*, Eur. J. Math., DOI: 10.1007/s40879-015-0076-y.
- [13] O. Karlova, V. Mykhaylyuk, *Functions of the first Baire class with values in metrizable spaces*, Ukr. Math. J. **58** (4) (2006), 567–571 (in Ukrainian).
- [14] O. Karlova, O. Sobchuk, *On H_1 -compositors and piecewise continuous mappings*, Mat. Studii **38** (2) (2012), 139–146 (in Ukrainian).
- [15] G. Koumoullis, *A generalization of functions of the first class*, Top. Appl., **50** (1993), 217–239.
- [16] K. Kuratowski, *Topology. Volume 1*, Academic Press (1966).
- [17] Peng-Yee Lee, Wee-Kee Tang, Dongsheng Zhao, *An equivalent definition of functions of the first Baire class*, Proc. Amer. Math. Soc., **129**(8) (2000), 2273–2275.
- [18] J. Jachymski, M. Lindner, S. Lindner, *On Cauchy type characterizations of continuity and Baire one functions*, Real Anal. Exchange, **30**(1) (2004/05), 339–346.
- [19] D. Lecomte, *How we can recover Baire class one functions?*, Mathematika **50**(1-2) (2003), 171–198.
- [20] D.N. Sarkhel, *Baire one functions*, Bull. Inst. Math. Acad. Sinica, **31**(2) (2003), 143–149.
- [21] V. Mykhaylyuk, *Baire classification of pointwise discontinuous functions*, Nauk. visnyk Cherniv. un-tu. Matematyka. **76** (2000), 77–79 (in Ukrainian).
- [22] D. Zhao, *Functions whose composition with Baire class one functions are Baire class one*, Soochow J. Math., **33**(4) (2007), 543–551.

DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS AND INFORMATICS, YURI FEDKOVYCH CHERNIVTSI NATIONAL UNIVERSITY, KOTSUBYBNS'KOHO STR., 2, CHERNIVTSI, UKRAINE